On Reliability, Risk, and Gambling

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Abstract

In this paper, we explore the analytical relationship between reliability, risk, and gambling, with respect to engineering decisions. In particular, we review the tenets of rational gambling and their consequence: the expected utility theorem. This theorem leads naturally to an optimization formulation the solution of which identifies that decision alternative having the most favorable risk. We formalize the relationship between risk and reliability, and thus the relationship between reliability and gambling. The results given here offer principles from which analytical methods can be developed. These normative principles can also be used as a litmus test for evaluating the validity of new or existing methods and analyses.

1 Introduction

Engineers are professional gamblers, making risk encumbered decisions addressing the design, development, deployment, and operation of technology. Yet, most engineers are unacquainted with the tenets of *gambling*. In the sections that follow, we will review the normative foundations of decision making, and we will expose the analytical relationship that exists between reliability, risk, and gambling.

It is *not* our purpose to explore decision support methodologies; rather, our focus is given to acquainting the reader with principles from which useful analytical methods can be developed. These principles can also be used as a litmus test for evaluating the validity of new or existing methods and analyses. The proofs of well know results are left to cited references.

Our exposition of principles is intended for readers having expertise in reliability engineering, and it is our hope that those readers who are unfamiliar with the relationship between reliability, risk, and gambling can use this paper to "put a stake in the ground," and move forward towards a more complete understanding.

The purpose of *reliability theory* is to support improving the reliability of technology. Reliability theory yields, of course, engineering models, and these models typically support making engineering decisions. The modern definition of reliability is given in the language of probability theory [**bar75**], quantifying the uncertainty of "lifetime". Risk quantifies uncertainty of the value associated with a particular decision alternative. Gambling is, for our purposes, the act of making rational engineering decisions (typically with the support of models derived from risk and reliability theory). Hence, gambling (in engineering) is necessarily an analytical exercise where one chooses the "most favorable" from among available risk encumbered alternatives.

The words *reliability, risk, and gambling* are widely used in casual conversation, where considerable license is allowed as to their respective meanings. We shall, however, require precise terminology[1].

- **Reliability** is the probability law on the lifetime of a particular device or system.

- **Risk** is the probability law on the anticipated monetary value[2] of a particular decision alternative.

- **Gambling** is the act of *rationally choosing* the "most favorable" from among all available risk encumbered decision alternatives.

Each of these definitions asserts *uncertainty*; hence, their analytical characterizations necessarily derive from probability theory[3]. Inasmuch as our definition of gambling requires rational behavior, our analytical characterizations are also built upon normative principles (in the form of axioms) that we shall establish.

Observation 1 *Gambling is an exercise in optimization, requiring an objective function*

[1]There are a great many stylized definitions of risk. In engineering application, for example, risk is often characterized as the probability of some undesirable event. In this paper, we employ the definition of risk prefered by most economists [**mar˙87**]; a bit of careful thought will reveal that this definition readily subsumes most characterizations of risk appearing in the engineering literature.

[2]Generally, it is possible for this value to be negative.

[3]Additivity (more specifically countable–additivity) of measure is essential in order to avoid pitfalls such as arbitrage.

accompanied by constraints. This objective function must *be formulated, functionally, in terms of risk. In engineering applications, risk is often written as a function of reliability.*

Historically, the reliability literature has given much attention to understanding uncertainties associated with the anticipated performance of technology. Whether arising as decisions addressing design or operation, gambling is an inescapable activity of the engineering endeavor[4]; yet, the reliability literature is largely disconnected from this activity. The connection between technology performance (*i.e.,* reliability) and engineering decisions (*i.e.,* gambling) is made through *risk* – the characterization of value under uncertainty. Thus, a hierarchy is established for engineering decisions: In order to quantify gambling, one must first quantify risk; in order to quantify risk, one often quantifies reliability.

2 Tenets of Gambling

The tenets of gambling are normative and thus axiomatized. The axiomatic framework that we present is not new; our exposition summarizes well–known results that have been reported broadly in the mathematics and mathematical economics literatures. The origins of our exposition can be credited to von Neumann and Morgenstern [**von47**].

We shall review the axioms defining rationality and their consequence – the *expected utility theorem* – and comment on their appropriateness for application to engineering [5].

[4]Nearly all engineering decisions *must* address value in the presence of uncertainty.

[5]There are several alternative axiomatic formulations leading to expected utility theorems, see for example [**sav54**], similar to von Neumann – Morgenstern. While we accept von Neumann – Morgenstern,

It is important to recognize that the expected utility theorem provides the basic principle on which all decision/risk theoretic methods are based. The theorem guarantees that, for a rational gambler, there exists a function (unique up to affine transformations) that will induce a measure assigning a numerical value to each risk encumbered decision alternatives such that preferences among alternatives follows a numerical ordering. A gambler who violates any of the underlying axioms is said to be irrational.

The expected utility theorem is essential because it provides an assessment of value based on *ordinal* preferences. The von Neumann – Morgenstern axioms [**von47**] establish that the consequence of any decision is a (possibly negative) reward, mapped into a common unit of exchange (*e.g.,* money). In our discussions, we will connect reward (value) with "physics" (reliability).

Consider a gambling opportunity where a gambler must select a single alternative from among the available alternatives, where $\mathcal{A}$ is the set of indices for the alternatives[6]. Without loss of generality, let $\mathbb{A} = [a, b] \subset \mathbb{R}$ be a compact interval on the real line containing the point 0. Thus, $\mathbb{A}$ a closed and bounded set containing all possible dollar rewards associated with a gambling opportunity. Here, b is the maximum possible reward and 0 the *status quo* – in practice, gambling alternatives take only finite values.

The *risk* associated with a particular gambling alternative is defined as the cumulative probability distribution function of that alternative's reward. That is, each decision alternative $\alpha \in \mathcal{A}$ has a corresponding reward distribution function $F_\alpha(\cdot) \in D(\mathbb{A})$. Here,

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other axiomatic formulations would be equally acceptable for our purposes.

[6]The index set $\mathcal{A}$ must contain at least two alternatives. When there are exactly two alternatives, the gambling opportunity is a *take it or leave it* bet.

elements of $D(\mathbb{A})$ are right-continuous, non-decreasing functions mapping $\mathbb{A}$ into the unit interval (*i.e.,* all distributions having support that is a subset of $\mathbb{A}$). For each decision alternative there is a probability law (*i.e.,* distribution) governing the amount of reward to be gained by selecting that alternative.

Recall that the objective of gambling is to select from among all available alternatives that alternative having the most favorable risk; that is, to choose the alternative having the most preferred reward distribution $F_\alpha^* \in D(\mathbb{A})$. The expected utility theorem and its supporting axioms [**pup91**] offer a foundation for building methodologies that accomplish this objective.

Axiom 1 *(Weak Ordering) There is a preference relation* $\succcurlyeq$ *among the elements of* $D(\mathbb{A})$ *that is complete and transitive.*

That is, a gambler (engineer) comparing any two alternatives (reward distributions) should prefer one over the other[7], and his preferences among all reward distributions are transitive. It is argued (and we agree) that transitivity is characteristic of rational behavior. Transitivity reflects self–consistency[8].

Axiom 2 *(Continuity). For every set* $F \in D(\mathbb{A})$ *the sets* $\{G \in D(\mathbb{A}) : G \succcurlyeq F\}$ *and* $\{G \in D(\mathbb{A}) : F \succcurlyeq G\}$ *are closed in the topology of weak convergence.*

[7] Weak orderings allow indifference among alternatives, *i.e.,* preferences are not necessarily strict.

[8] However, self–consistency is not necessarily easily achieved. If, for example, you were given graphs of the reward distributions for, say 300, alternatives of a given gambling opportunity, arranging these graphs in the order of your preference for the alternatives could be quite difficult – even though these distributions carry complete information on the uncertainty associated with the gambling opportunity.

That is, for any sequence of distributions $\{F_1, F_2, ...\} \in D(\mathbb{A})$ that converges at all points of continuity to a distribution F it is required that: (1) If each F_n is preferred to G, then F is preferred to G, and (2) if G is preferred to each F_n then G is preferred to F. It is argued that rational behavior does not allow one to exclude the limiting distribution F from our preference ordering.

Axiom 3 *(Independence) For all $F, G, H \in D(\mathbb{A})$ and all $\lambda \in [0,1]$, $F \succcurlyeq G$ implies $\lambda F + (1-\lambda)H \succcurlyeq \lambda G + (1-\lambda)H$.*

The normative appeal of this axiom lies with its interpretation under compound lotteries, where ones preferences within a given lottery should be independent of previous lottery outcomes. Generally, this axiom asserts that, if reward distribution F is preferred to distribution G, then this preference is unchanged over mixtures of F and G, respectively, with a third distribution H.

Theorem 1 *(Expected Utility Theorem.) Let $\succcurlyeq$ be a binary relation on $D(\mathbb{A})$. There exists a continuous function $u : \mathbb{A} \to \mathbb{R}$ (unique up to affine transformations) such that $F \to \int_{\mathbb{A}} u(x)dF(x)$ represents $\succcurlyeq$ if and only if Axioms 1, 2, and 3 are satisfied.*

Observation 2 *Clearly, utility is a function that separates points in the set of distribution functions having support in $\mathbb{A}$. The expected utility theorem asserts that, when the three axioms defining rational behavior are satisfied, there exists a function u (the utility function) that can be used to reveal one's preferences among risk encumbered alternatives.*

Decisions (gambles) are made by individuals [**haz˙96**] (not groups)[9]; hence, utility is

[9]The interactions among a group of individuals, who's respective decisions are influenced by the decisions of other group members is the focus of *game theory.*

specific to an individual gambler at the epoch of a decision[10]. The utility function first appeared in the literature in [**ber89**]; however, it was not until [**von47**] that utility was axiomatized.

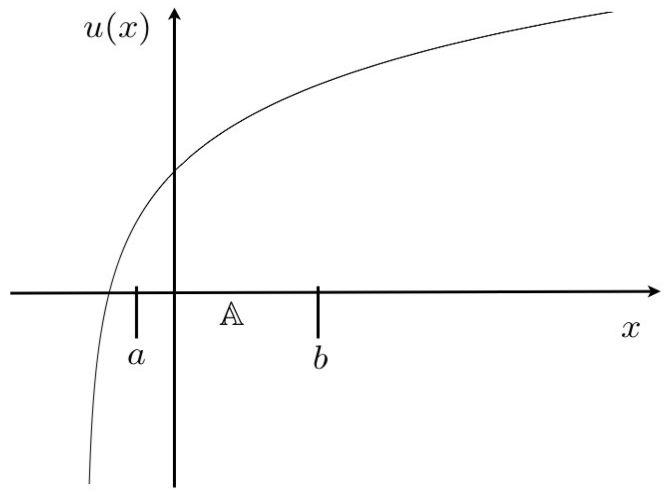

Figure 1: Hypothetical utility function of a risk averse gambler.

Figure (1) shows the graph of a hypothetical utility function, revealing how a gambler values money at the epoch of a hypothetical gambling opportunity. Because the utility function is unique up to affine (linear) transformation, utility separates risk encumbered alternatives independent of the monetary units chosen. Note that, in this example, the gambler holds a diminishing marginal value for money and a sharp distaste negative wealth. There are numerous well–documented procedures for capturing an individual's utility function; we shall not address them here.

[10]As is revealed by Arrow's Impossibility Theorem [**arr**], group utility functions cannot in general be constructed.

Gambling boils down to the following basic idea: Beginning with a utility function and a set of reward distributions associated with risky decision alternatives, search among the alternatives for the one having the greatest expected utility. That is , for each $\alpha \in \mathcal{A}$ there is a unique reward distribution $F_\alpha \in D(\mathbb{A})$; the most favorable alternative α^* is determined by

$$\alpha^* = \arg\max_{\alpha \in \mathcal{A}} \int_{\mathbb{A}} u dF_\alpha, \tag{1}$$

$$s.t.$$

$$F_\alpha \in D(\mathbb{A}).$$

Clearly, α^* need not be unique.

It is straightforward to develop an alternate formulation of (1) that provides additional insight into rational gambling. It follows as corollary [**von47**] to the expected utility theorem that u is continuous and nondecreasing. Hence, u induces a measure on $(\mathbb{R}, \mathcal{B})$. Further, since each element of $D(\mathbb{A})$ is a distribution with compact support $\mathbb{A}$, expected utility can be expressed through a change of measure via integration by parts as

$$\int_a^b u dF_\alpha = u(b) - \int_a^b F_\alpha du, \forall F_\alpha \in D(\mathbb{A}).$$

The integral $\int_a^b F_\alpha du$, $F_\alpha \in D(\mathbb{A})$, measures risk F_α (with respect to a gambler's utility u for money) for a particular alternative $\alpha \in \mathcal{A}$ in a gambling opportunity. It now follows that optimization formulation (1) can be rewritten as

$$\alpha^* = \arg\min_{\alpha \in \mathcal{A}} \int_a^b F_\alpha du, \tag{2}$$

$$s.t.$$

$$F_\alpha \in D(\mathbb{A}).$$

Formulation (2) reveals that a rational gambler will, when presented with the alternatives of a gambling opportunity, search for that alternative presenting the *least measure of risk*.

The shape of a gambler's utility function reflects his appetite for risk: when u is concave over $\mathbb{A}$, a gambler is said to be *risk averse*; when u is convex over $\mathbb{A}$, he is said to be *risk perverse*; when u is linear over $\mathbb{A}$, he is said to be *risk neutral*. It is often the case that the alternatives of a gamble are such that the reward interval $\mathbb{A}$ is sufficiently small that u becomes (approximately) linear; in such case, u induces Lebesgue measure. Under Lebesgue measure (*i.e.,* risk neutrality), optimization formulations (2) and (1) seek an alternative having the *greatest expected reward*[11].

3 The Relationship Between Risk and Reliability

We now focus our attention on the relationship between risk and reliability. Costs and revenues associated with engineering decisions cannot be predicted with certainty; hence, gambling opportunities frequently arise in the design and/or operation of a device or system. Clearly, *risk* (probability law on the value of an alternative) is defined only at

[11] Linear utility is commonly assumed in the formulation of Markov decision processes and other stochastic optimal control paradigms where the reward interval associated with control policy is small.

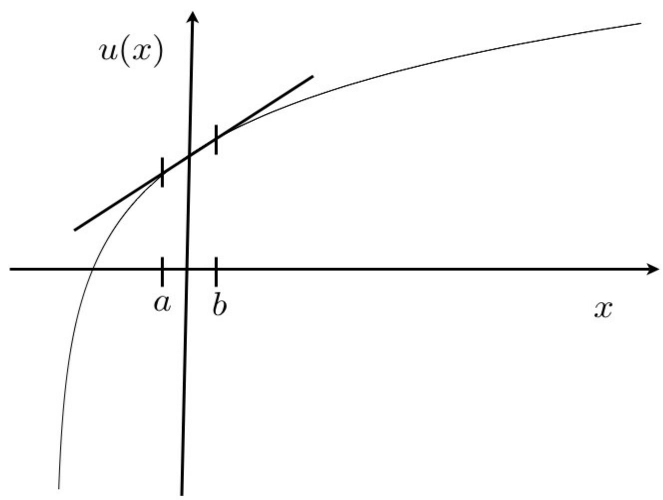

Figure 2: Hypothetical utility function with a *risk neutral* reward interval.

the epoch of decision in a gamble, whereas *reliability* (the probability law on device or system lifetime) reflects temporal behavior; the connection between these two laws is revealed through certain stochastic processes underlying them both.

In our discussions, it is unnecessary to distinguish between devices and systems (a device may be thought of as a system consisting of a single element); hence, we will enlist the word "system" when referring anything that might degrade and/or fail. We shall, for any particular system, define a *state process*. This stochastic process captures not only the physical state of the system but also the random environment (*i.e.*, thermodynamic, electromagnetic, economic, *etc.*) in which it operates. Thus, the state process is particular to any maintenance policies, control laws, financial strategies, regulatory restrictions, *etc.* that may in part govern system operation. In general, the state process may also capture the design, development, and deployment disposition of a system over time[12]. Thus, for any gambling opportunity $\alpha \in \mathcal{A}$, there is a *system state process* $\mathcal{X}_\alpha = \{X_\alpha(t); t \geq 0\}$ defined on the probability space $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$[13]. Here, $X_\alpha(t) : (\Omega_\alpha, \mathcal{F}_\alpha) \mapsto (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), t \geq 0$. In practical modeling scenarios, we typically regard the trajectories of $\mathcal{X}_\alpha$ to be of bounded variation *almost surely*.

The physics captured by $\mathcal{X}_\alpha$ is mapped to reward through a *reward function* $r_\alpha : ((\mathbb{R}^n \times \mathbb{R}_+), \mathcal{B}((\mathbb{R}^n \times \mathbb{R}_+))) \mapsto ((\mathbb{R} \times \mathbb{R}_+), \mathcal{B}((\mathbb{R} \times \mathbb{R}_+)))$. Here, $r_\alpha(x, t)$, $x \in \mathbb{R}^n, t \geq 0$ is the (possibly negative) *rate of reward* received under gamble alternative α when the system is in state x at time t. We shall allow r_α to include any discounting directing the time value of money. We shall impose the practical restrictions that r_α be bounded, of finite variation, and vanishing beyond some finite time; hence, r_α is integrable. With $Y_\alpha(t)$ taken as the total reward accumulated under alternative α by time t, we have that

$$Y_\alpha(t) = \int_0^t r_\alpha(X_\alpha(s), s) ds,$$

and

$$\lim_{t \to \infty} |Y_\alpha(t)| < \infty.$$

It now follows that value of alternative α is given by

$$V_\alpha = \lim_{t \to \infty} Y_\alpha(t) = \int_0^\infty r_\alpha(X_\alpha(s), s) ds, \tag{3}$$

[12]System state processes are typically very difficult to analytically characterize.

[13]It is not necessary that the various alternatives of a gamble to be defined on the same probability space.

and the risk associated with α is

$$F_\alpha(v) = P_\alpha\{V_\alpha \le v\} = P_\alpha\{\int_0^\infty r_\alpha(X_\alpha(s), s)ds \le v\}. \qquad (4)$$

Observation 3 *For a particular alternative of a gambling opportunity, where a system operates within an alternative–specific (random) environment, equation (4) gives the relationship between risk and the evolution of system state. One practical implication of this relationship is that risk associated with technology must incorporate lifecycle reward – from the begining of system design onward.*

We note that equation (4) does not provide a direct relationship between system reliability and risk. In order to establish this relationship, we must first formalize the relationship between system state and system reliability.

Let $B \subset \mathcal{B}(\mathbb{R}^n)$ denote the set of states where the system under consideration is operational. With $1_B(\cdot)$ taken as the indicator function on the set B, it is clear that $E_\alpha(1_B(X_\alpha(t)) = P_\alpha\{X_\alpha(t) \in B\}$ is the probability that the system is operational at time t. We shall refer to the stochastic process $\mathcal{Z}_\alpha = \{1_B(X_\alpha(t)), t \ge 0\}$ as the *reliability process*. In situations where B is an absorbing state (*e.g.*, the system cannot recover from failure), we have that $T_\alpha = \inf\{t \ge 0; 1_B(X_\alpha(t)) = 0\}$ is the system lifetime and $R_\alpha(t) = P_\alpha\{T_\alpha > t\} = E_\alpha(1_B(X_\alpha(t)))$ is the system *reliability*. When B is not an absorbing state, $E_\alpha(1_B(X_\alpha(t)))$ gives the system *availability* at time t, and $\lim_{t\to\infty} E_\alpha(1_B(X_\alpha(t)))$ is called the *limiting availability*.

Observation 4 *The usefulness of $\mathcal{Z}_\alpha$ arises because it is typically not possible to directly observe the state process $\mathcal{X}_\alpha$. However, it is often feasible to determine, for all $t \ge 0$,* *whether or not a system is in a operational state – in which case $\mathcal{Z}_\alpha$ is easily observed. Thus, the reliability process $\mathcal{Z}_\alpha$ offers a coarse surrogate for the complete characterization of system state given in $\mathcal{X}_\alpha$.*

In practice, the functional relationship between $\mathcal{Z}_\alpha$ and value can be difficult to established and requires the introduction of a filtered probability space. Consider the filtered probability space $(\Omega_\alpha, \mathcal{F}_\alpha, \{\mathcal{F}_\alpha(t); t \geq 0\}, P_\alpha)$ which is our usual probability space augmented with the filtration $\{\mathcal{F}_\alpha(t); t \geq 0\}$ to which $\mathcal{Z}_\alpha$ will be adapted. Here, $\{\mathcal{F}_\alpha(t); t \geq 0\}$ is a collection of sub–σ–algebras of $\mathcal{F}_\alpha$ such that $\mathcal{F}_\alpha(s) \subset \mathcal{F}_\alpha(s+t), \forall s, t \geq 0$, and $\mathcal{F}_\alpha(\infty) = \lim_{t\to\infty} \mathcal{F}_\alpha(t) \subset \mathcal{F}_\alpha$. Here, $\{\mathcal{F}_\alpha(t); t \geq 0\}$ is the (stochastic) *history*[14] of the the reliability process $\mathcal{Z}_\alpha$, with $\mathcal{F}_\alpha(t) = \sigma(\{1_B(X_\alpha(u); u \leq t\})$ is the sub–σ–algebra generated with $\mathcal{Z}_\alpha$ truncated to t. Hence, we have that $\mathcal{F}_\alpha(\infty) = \sigma(\mathcal{Z}_\alpha)$ is the σ–algebra (*i.e.,* event set) generated by the reliability process.

For $t \geq 0$, let $G_{Y_\alpha(t)}$ be the distribution of total accumulated reward under alternative α by time t. We now formulate the relationship between accumulated reward and reliability as follows:

$$G_{Y_\alpha(t)}(y) = P_\alpha\{Y_\alpha(t) \leq y\} = E_\alpha(P_\alpha\{Y_\alpha(t) \leq y\} | \mathcal{F}_\alpha(t)). \tag{5}$$

That is, the distribution of the total reward, under alternative α, accumulated by time t depends on the *history* of the reliability process $\mathcal{Z}_\alpha$ up to and including time t. It now follows from equation (3) and bounded convergence theorem that the relationship between

[14]The filtration $\{\mathcal{F}_\alpha(t); t \geq 0\}$ captures *observable information* associated with system dynamics.

risk F_α and reliability is given by

$$\begin{aligned} F_\alpha(y) &= \lim_{t\to\infty} G_{Y_\alpha(t)}(y) \\ &= E_\alpha(\lim_{t\to\infty} P_\alpha\{Y_\alpha(t) \le y\}|\mathcal{F}_\alpha(t)\}). \end{aligned} \tag{6}$$

Note that in equation (6), for every $t \geq 0$, each version of $P_\alpha\{Y_\alpha(t) \le y\}|\mathcal{F}_\alpha(t)\}$ is a random variable and thus, almost surely,

$$\lim_{t\to\infty} P_\alpha\{Y_\alpha(t) \le y|\mathcal{F}_\alpha(t)\} = P_\alpha\{V_\alpha \le y|\mathcal{F}_\alpha(\infty)\}.$$

It now follows that

$$\begin{aligned} F_\alpha(y) &= E_\alpha(P_\alpha\{V_\alpha \le y|\mathcal{F}_\alpha(\infty)\}) \\ &= E_\alpha(P_\alpha\{\int_0^\infty r_\alpha(X_\alpha(s), s)ds \le y|\sigma(\mathcal{Z}_\alpha)\}). \end{aligned} \tag{7}$$

Equation (7) reveals the functional relationship between risk, system state, and reliability.

4 Gambling

We now revisit gambling, where our objective is to select the alternative α^* having the most favorable (*i.e.*, minimum measure with respect to utility of) risk. Taking optimization formulation (2) and equation (7) together, we have that

$$\begin{aligned} \alpha^* &= \arg\min_{\alpha\in\mathcal{A}} \int_a^b F_\alpha(y)du(y), F_\alpha(y) \in D(\mathbb{A}) \\ &= \arg\min_{\alpha\in\mathcal{A}} \int_a^b E_\alpha(P_\alpha\{\int_0^\infty r_\alpha(X_\alpha(s), s)ds \le y|\sigma(\mathcal{Z}_\alpha)\})u(dy) \end{aligned} \tag{8}$$

Observation 5 *Equation (8) establishes the relationship between reliability, risk, and gambling. It shows that, when presented with a gambling opportunity having decision alternatives indexed by the set $\mathcal{A}$, the most preferred alternative $\alpha^* \in \mathcal{A}$, holds the minimum risk according to utility measure u, where risk F_{α^*} is functionally determined by that alternative's state $X_{\alpha^*}(t)$ (and subordinately it's reliability process $\mathcal{Z}_{\alpha^*}$) directing the dynamics of reward $r_{\alpha^*}(\cdot, t)$.*

Equation (8) is not a panacea; it does *not* reveal any specific gambling (decision support) methodology. Equation (8) simply establishes the relationship between reliability, risk, and gambling; useful gambling methodologies will reflect this relationship. Typically, much effort is required to capture (or even approximate) probability law on any of the stochastic processes that govern system dynamics. Reward functions, too, are often difficult to "nail down". Depending upon the sophistication of system dynamics and the reward function, computation of the integrals appearing in equation (8) can be very challenging. Finally, when the cardinality of $\mathcal{A}$ the index set of decision alternatives is large (large, here, can be a modest finite number), optimization over $\mathcal{A}$ can become especially difficult.

The challenges of engineering reliable systems are many and great. Specifically the level of difficulty associated with developing high–fidelity domain models underlying any gamble cannot be overstated. Yet, equation (8) is a direct and unavoidable consequence of rational behavior and, so long as **Axioms 1, 2, and 3** are accepted, it will remain so.

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