# Reliability, Risk, and High-Stakes Wagering

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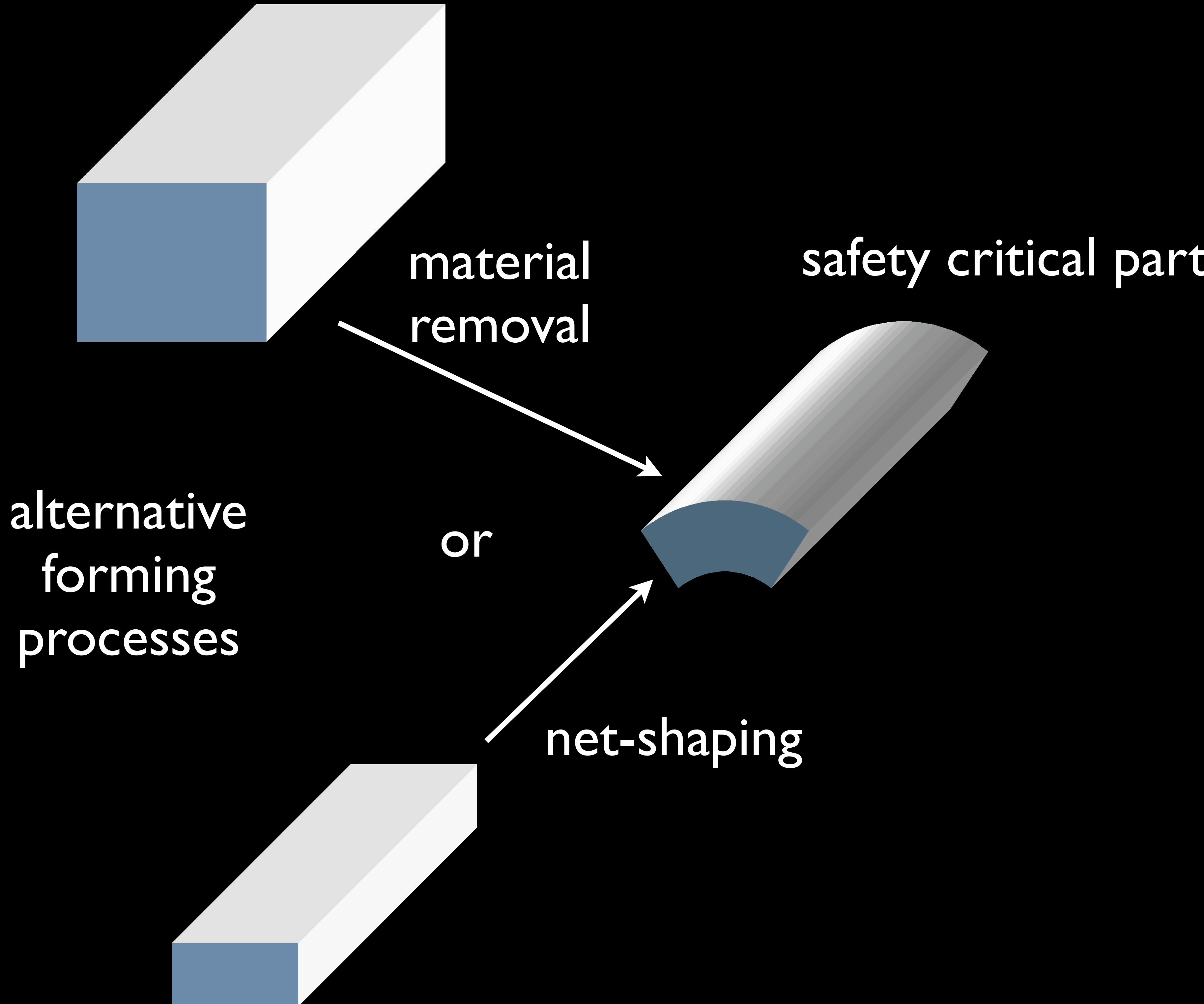
Prologue:

I am an engineer (not a philosopher, scientist, or mathematician).

Engineers are professional gamblers.

I make my living on the intersection of Reliability, Risk, and Gambling.

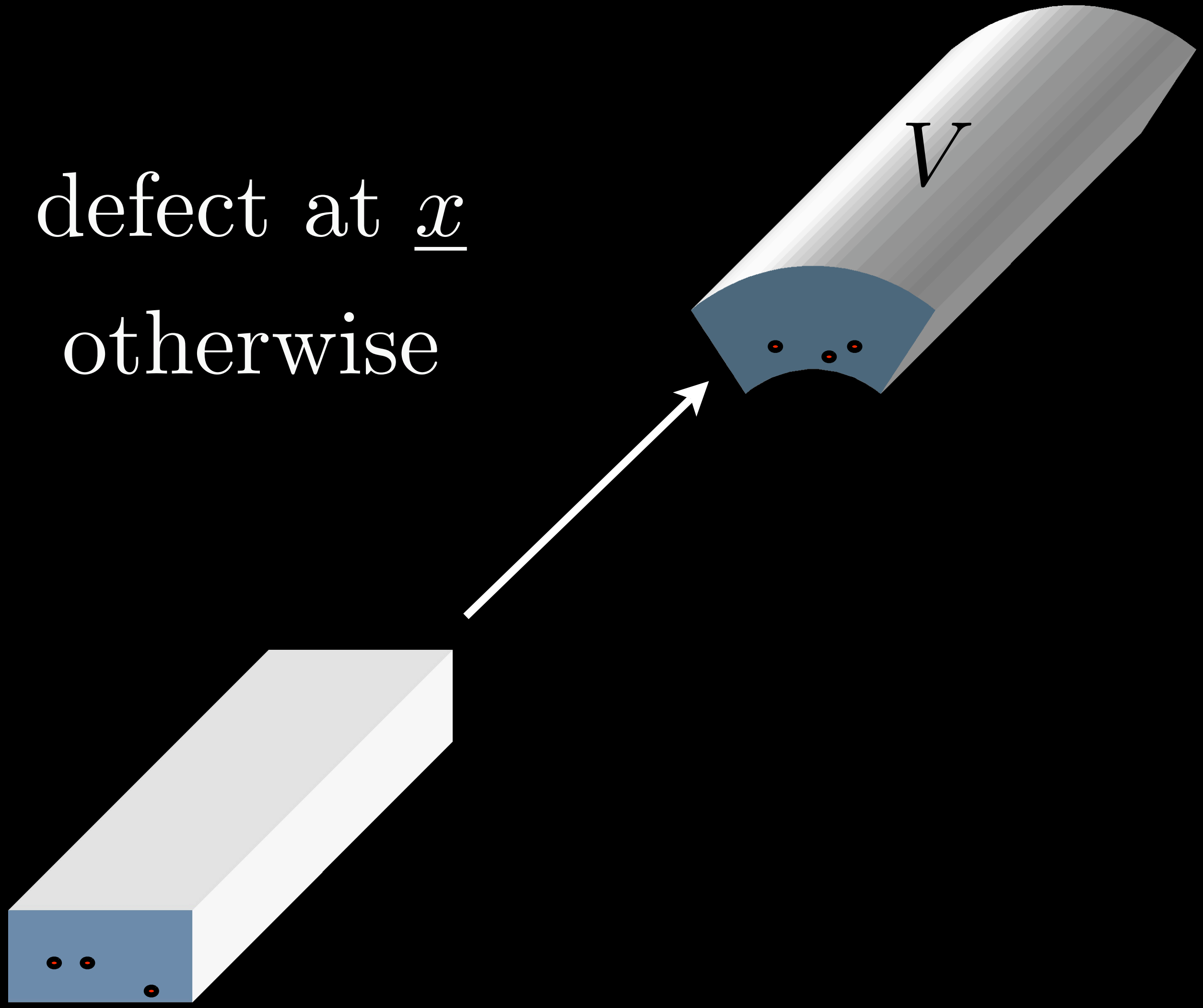
My research is focused on computational methods that support gambling (identifying engineered alternatives having the most favorable risk).


material
removal
safety critical part
alternative
forming
processes
or
net-shaping

# Point-defect mapping under net-shaping.

$$\{\varepsilon_V(\underline{x}); \underline{x} \in V \subset \mathbb{R}_+\}$$

$$\varepsilon_V(\underline{x}) = \begin{cases} 1 & \text{defect at } \underline{x} \\ 0 & \text{otherwise} \end{cases}$$

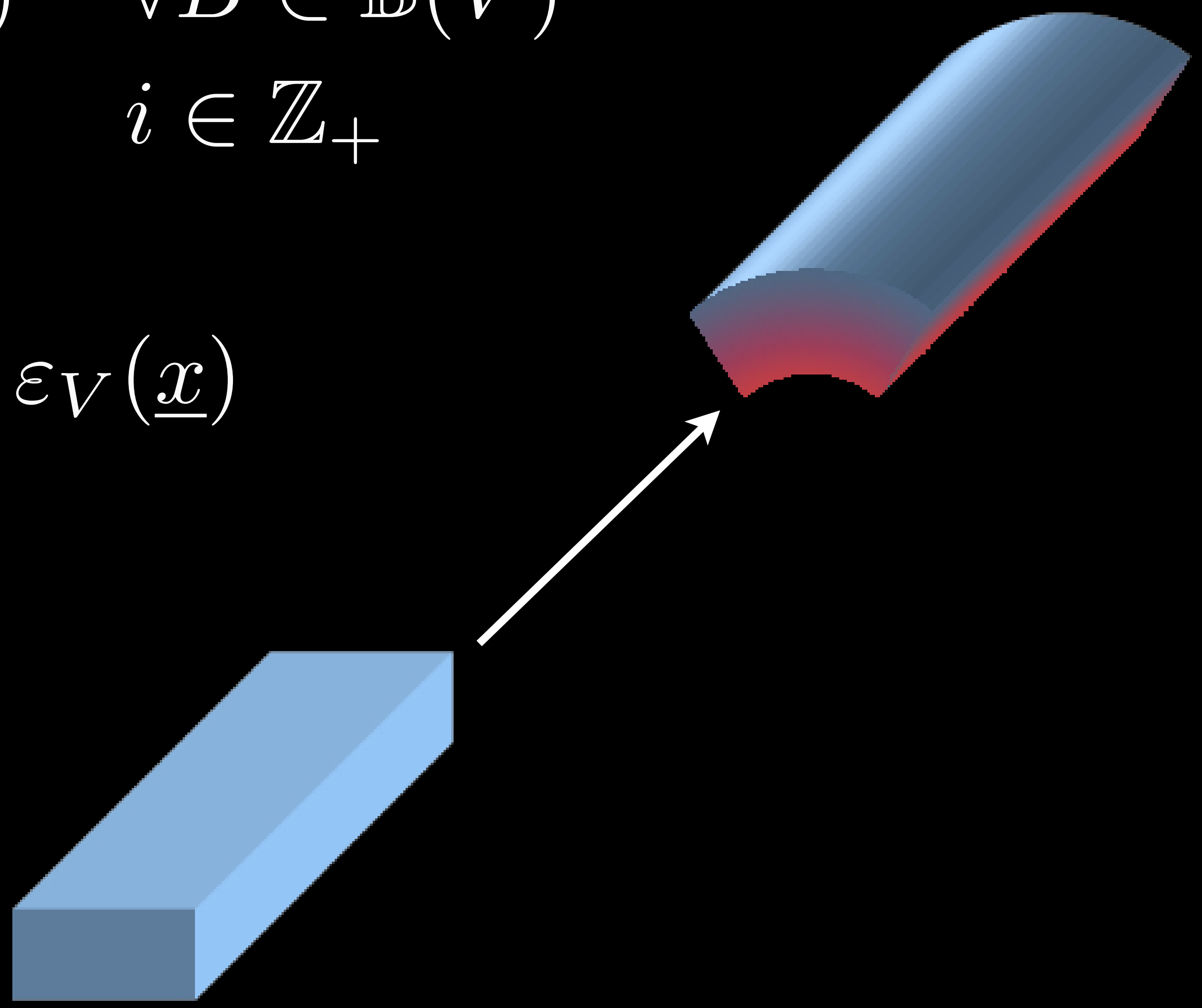


# Predictive Model:

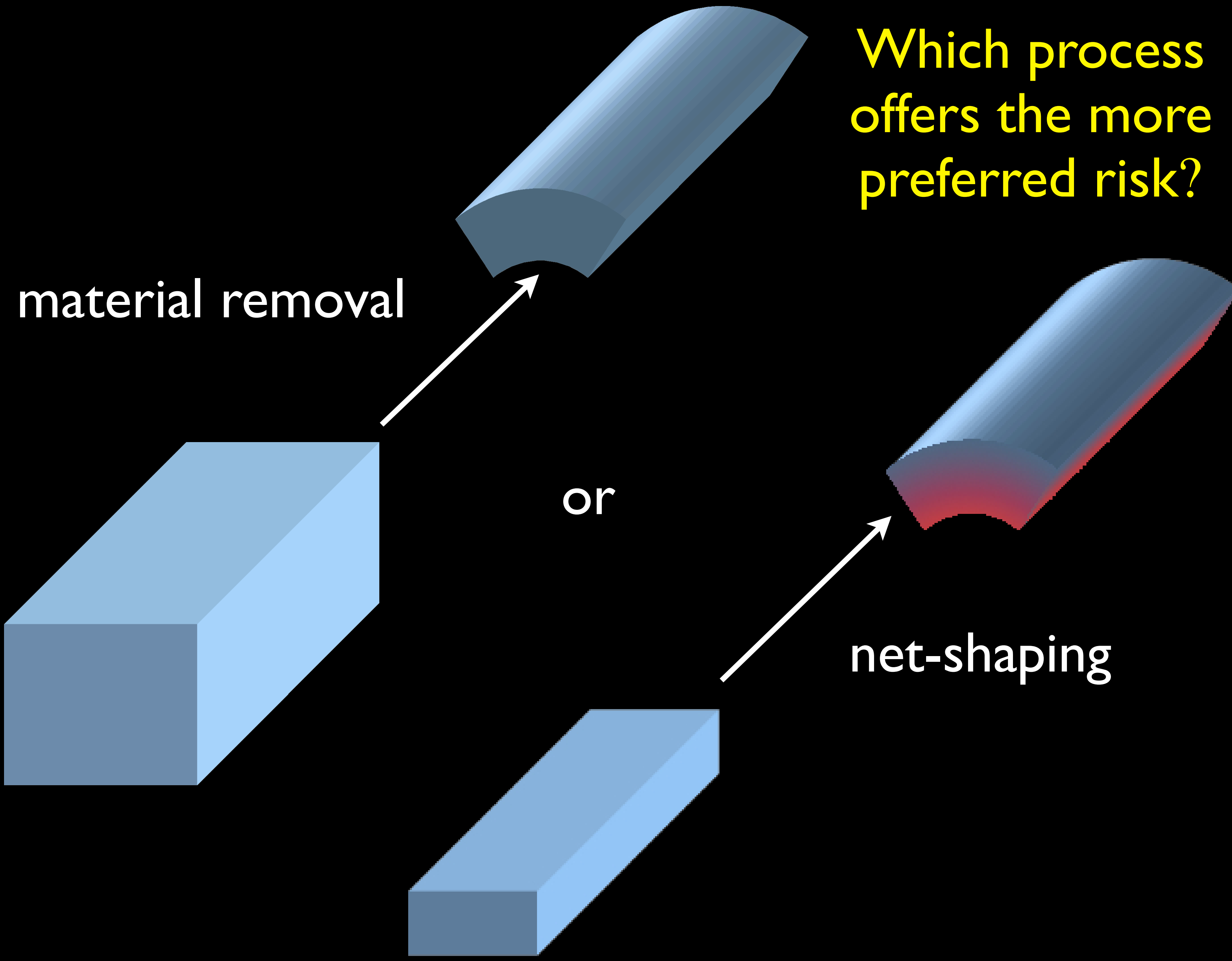
## Stochastic intensity of the point-defect probability law.

$$P(N_V(B) = i) \quad \forall B \in \mathbb{B}(V)$$
$$i \in \mathbb{Z}_+$$

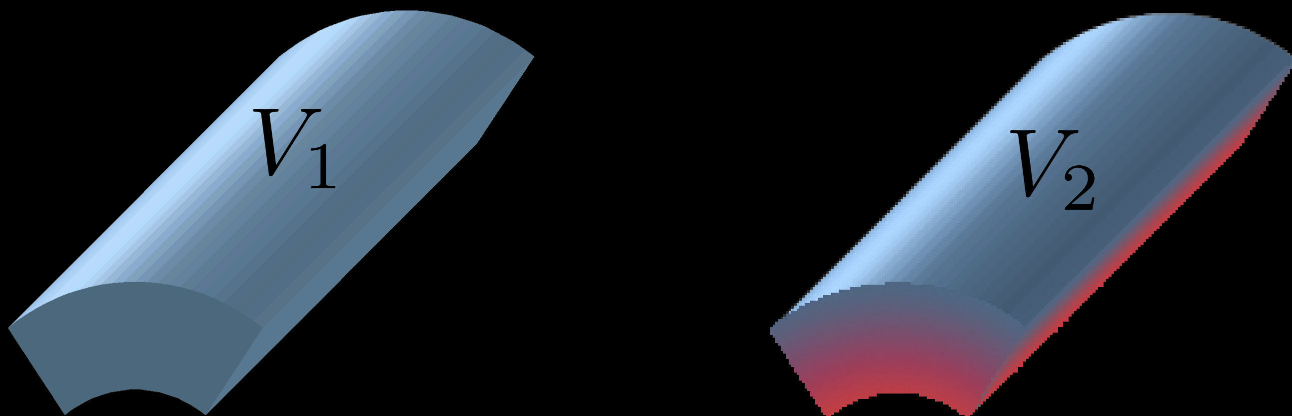
$$N_V(B) = \sum_{\underline{x} \in B} \varepsilon_V(\underline{x})$$



Which process offers the more preferred risk?



Value $V_\alpha$ is a random variable that is functionally related to the defect (stochastic) point process.




$$V_\alpha = g(\cdot, N_{V_\alpha}) \quad \alpha = 1, 2$$

# Present value of the reward marked point process.

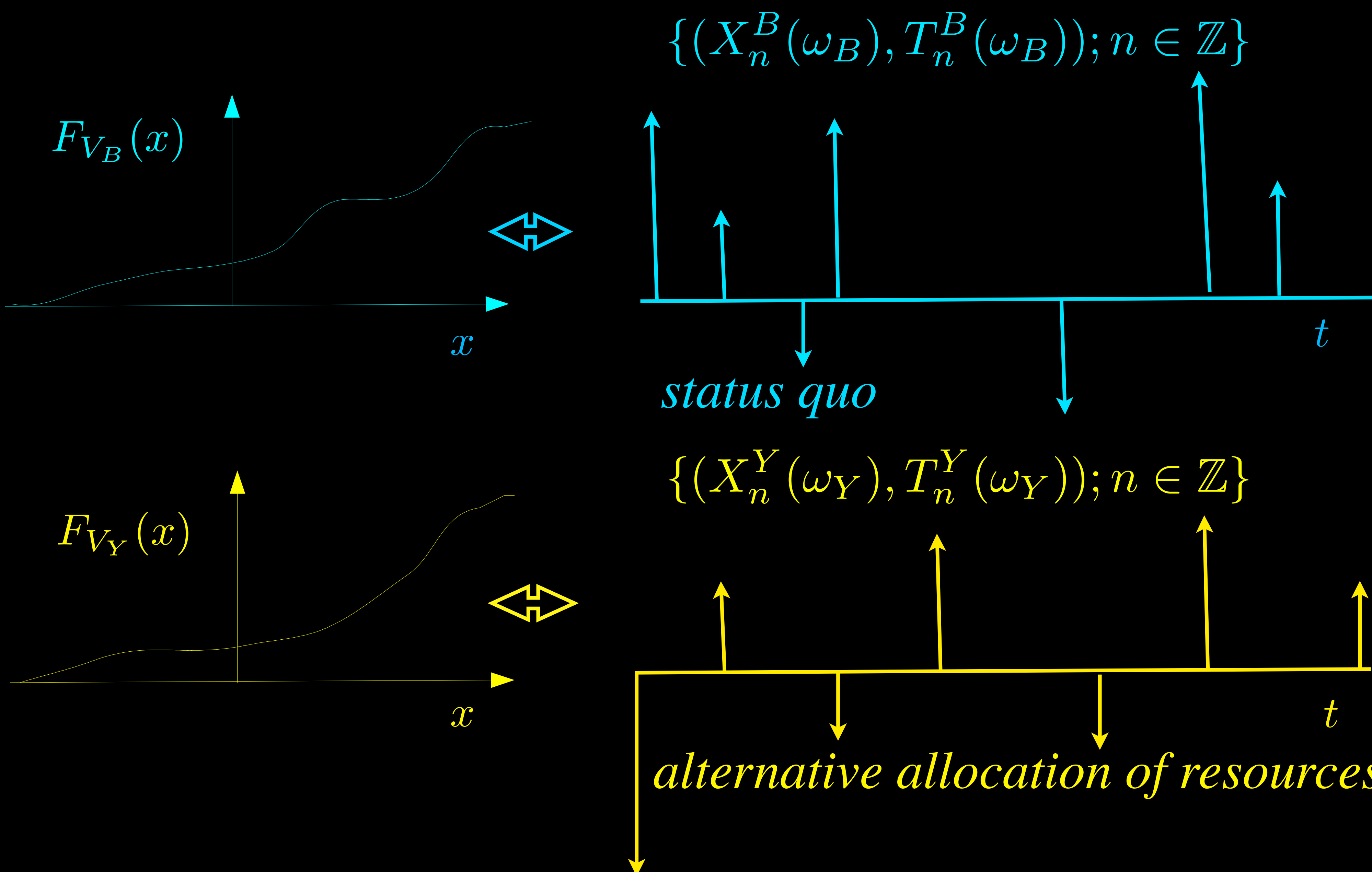
$$V = \sum_{n=0}^{\infty} f(X_n, T_n)$$

$f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ *discount function*

$$F_V(x) = P\{V \leq x\}$$

$F_V$ is called *risk* (a definition seemingly unfamiliar or misunderstood by all too many engineering professionals).

# Take-It-or-Leave-It Wagers



# The Alternatives of a Wager

Set of possible rewards.

$$\mathbb{A} = [a, b] \subset \mathbb{R}, 0 \in \mathbb{A}$$

$D(\mathbb{A})$ is the set of distributions having support in $\mathbb{A}$

Alternative $\alpha$ has risk $F_\alpha \in D(\mathbb{A})$

Note that the alternatives (marked point processes) associated with a wager need not be defined on a common probability space.

# Tenets of Wagering

*(Weak Ordering) There is a preference relation* $\succcurlyeq$ *among the elements of* $D(\mathbb{A})$ *that is complete and transitive.*

*(Continuity) For every set* $F \in D(\mathbb{A})$ *the sets* $\{G \in D(\mathbb{A}) : G \succcurlyeq F\}$ *and* $\{G \in D(\mathbb{A}) : F \succcurlyeq G\}$ *are closed in the topology of weak convergence.*

*(Independence) For all* $F, G, H \in D(\mathbb{A})$ *and all* $\lambda \in [0, 1]$ *,*
$F \succcurlyeq G$ *implies* $\lambda F + (1-\lambda)H \succcurlyeq \lambda G + (1-\lambda)H$*.*

When taken axiomatically, these tenets of normative wagering together with the axioms of probability measure define ***rational***.

# Separating Points in $D(\mathbb{A})$

Theorem

*Let* $\succsim$ *be a binary relation on* $D(\mathbb{A})$. *There exists a continuous function* $u : \mathbb{A} \to \mathbb{R}$ *(unique up to affine transformations) such that*

$F \mapsto \int_{\mathbb{A}} u dF$ *represents* $\succsim$ *if and only if the normative axioms of wagering are satisfied.*

Caution! This is an ordinal result. It can only reveal preferences among alternatives - not "how much preferred" one alternative is to another.

# Engineering meets Wagering

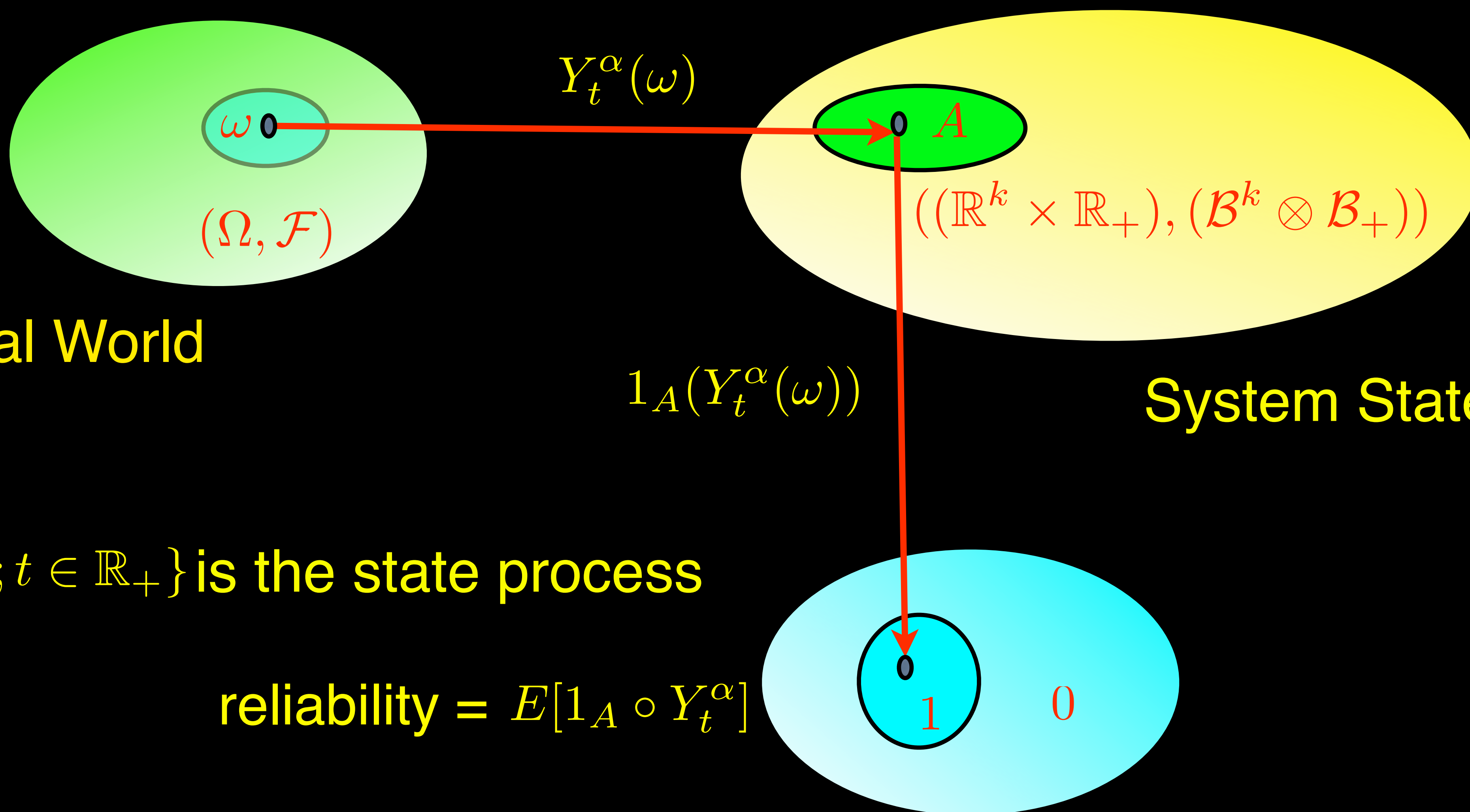
Engineering (e.g., design, operation, etc.) is all about synthesizing alternatives from which to choose. These alternatives form a family of processes $\{Y_t^\alpha; t \in \mathbb{R}_+\}, \alpha \in I$ .

Wagering is all about assigning value to each alternative and selecting the alternative having the most favorable risk

$$\alpha^* = argmax \int_{\mathbb{A}} u dF_\alpha$$

In order to wager in engineering, we must get in contact with the physics!

# The physics: enter reliability, etc.

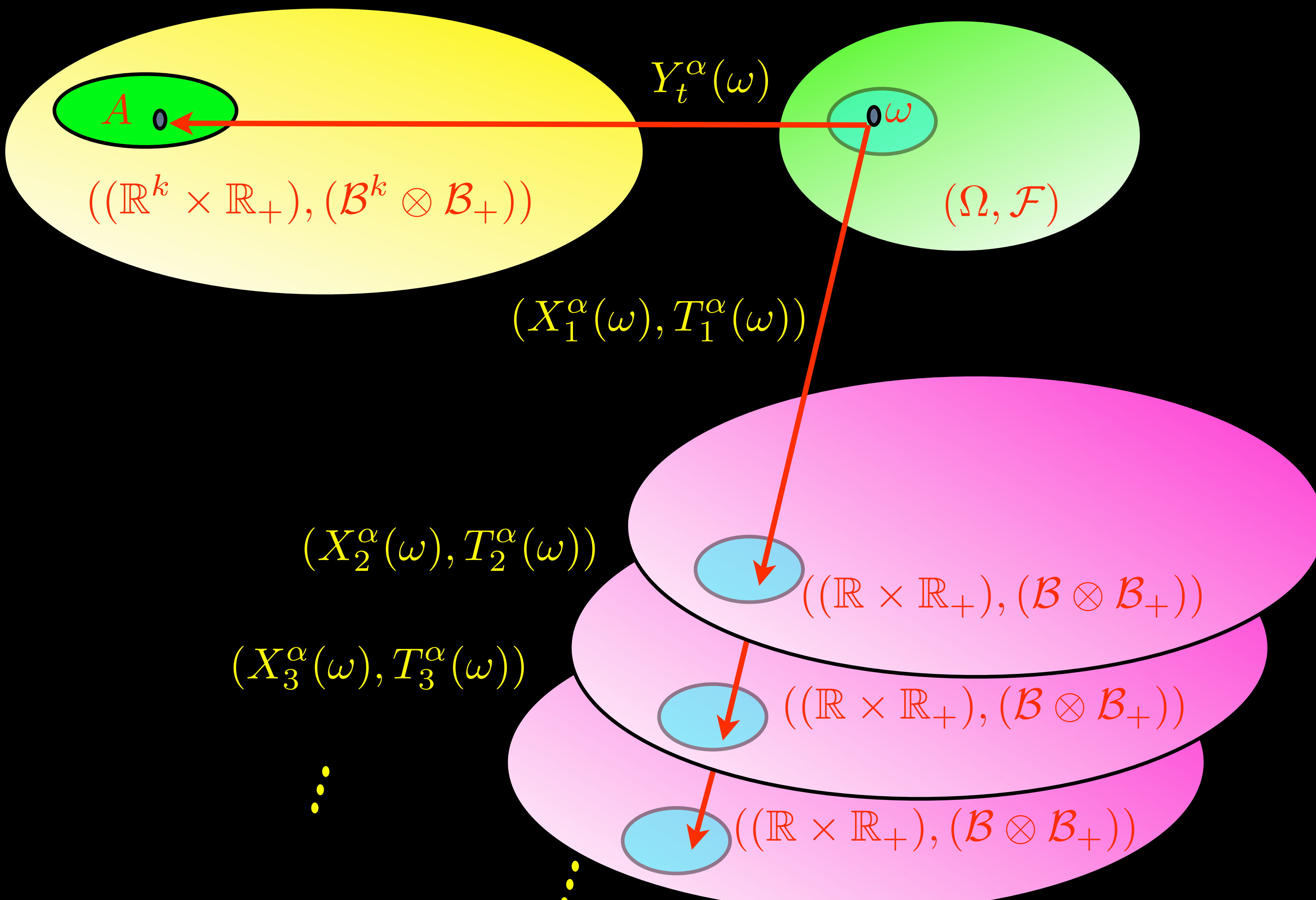



$\{Y_t^\alpha; t \in \mathbb{R}_+\}$ is the state process

reliability = $E[1_A \circ Y_t^\alpha]$

Rational wagering requires that we connect value with physics.

$\{(X_n^\alpha, T_n^\alpha); n \in \mathbb{Z}_+\}$ and $\{Y_t^\alpha; t \in \mathbb{R}_+\}$ are obviously dependent.



# So - here's the drill for wagering:

1) Go find the risk $F_\alpha$ for each alternative allocation of resources $\alpha$

2) At the moment prior to selecting an alternative, determine your "unique" $u : \mathbb{A} \rightarrow \mathbb{R}$

3) Compute $E_\alpha[u] = \int_{\mathbb{A}} u dF_\alpha$ for each $\alpha$

4) Choose the alternative $\alpha^* = argmax_\alpha E_\alpha[u]$

What could be simpler? (*Well - almost anything!*)

# What's the big deal, here?
# Why isn't wagering easy?

Because the risk of an alternative is the distribution on the present value of a random marked point process, one must use the probability law (i.e., all finite joint distributions) on $\{(X_n^\alpha, T_n^\alpha); n \in \mathbb{Z}_+\}$ in order to compute the corresponding risk. This is a staggering complicated computation.

Value of an alternative $V_\alpha = \sum_n f(X_n^\alpha, T_n^\alpha)$ is all about money - the technology is not explicitly represented, a disconnect with engineering.

High-stakes wagers are **never** concerned with repeatable experiments. Hence, you cannot appeal to an ergodic theorem to capture the probability laws on alternatives. These are one-and-off bets!

## Bad problem :-(

You will never have access to the probability laws on processes underlying a practical wager. Sorry.

## Pleasant observation :-)

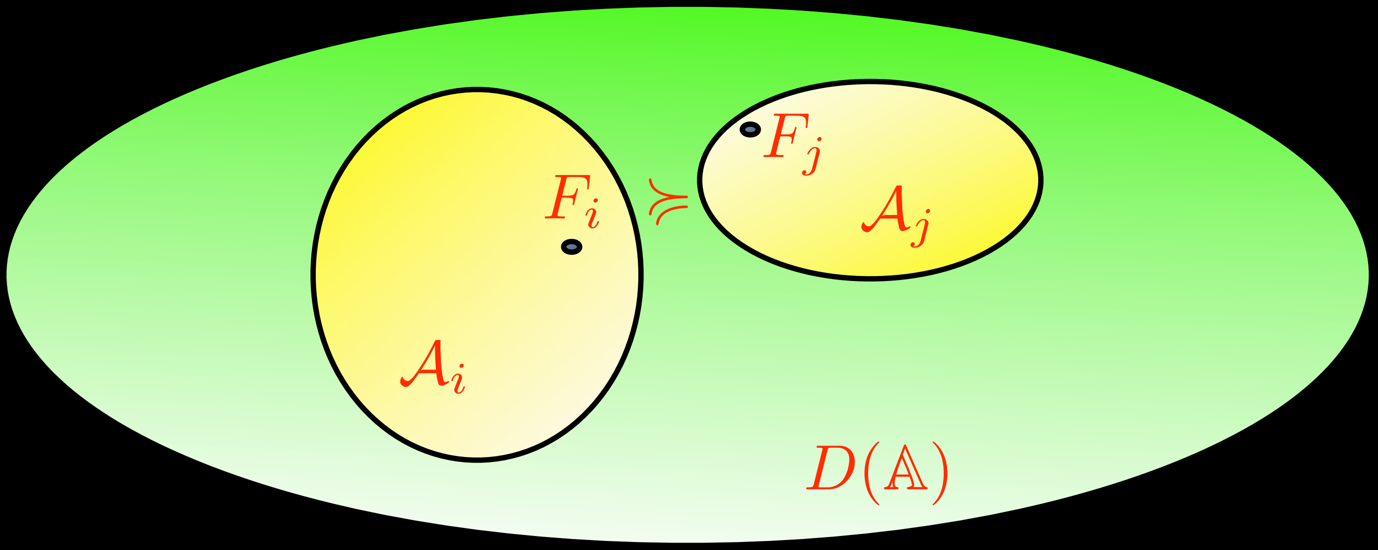
No one said that you have to explicitly know the risk of each alternative in order to select that alternative having the most favorable risk.

# Strategy

Connect with the engineering necessary to synthesis alternatives.

Capture each alternative's risk within some subset; $F_\alpha \in \mathcal{A}_\alpha \subset D(\mathbb{A})$. Here, the $\mathcal{A}_\alpha$ collection of distributions will be much simpler to characterize than the unique distribution $F_\alpha$ .

Appeal to the separation theorem to test $\mathcal{A}_i \succcurlyeq \mathcal{A}_j$ .

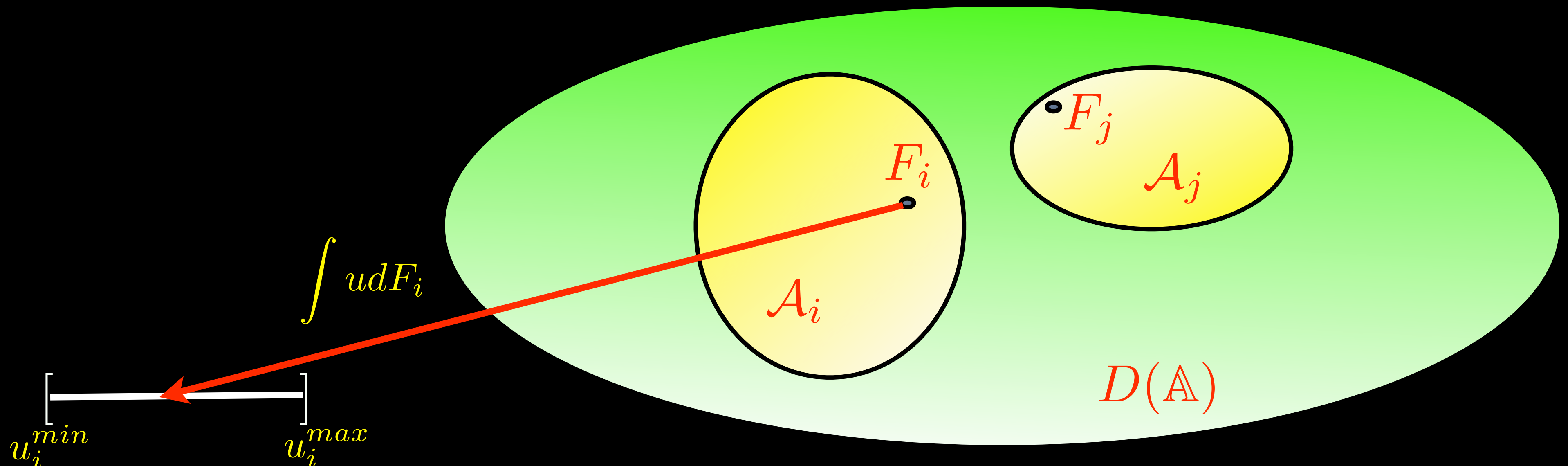


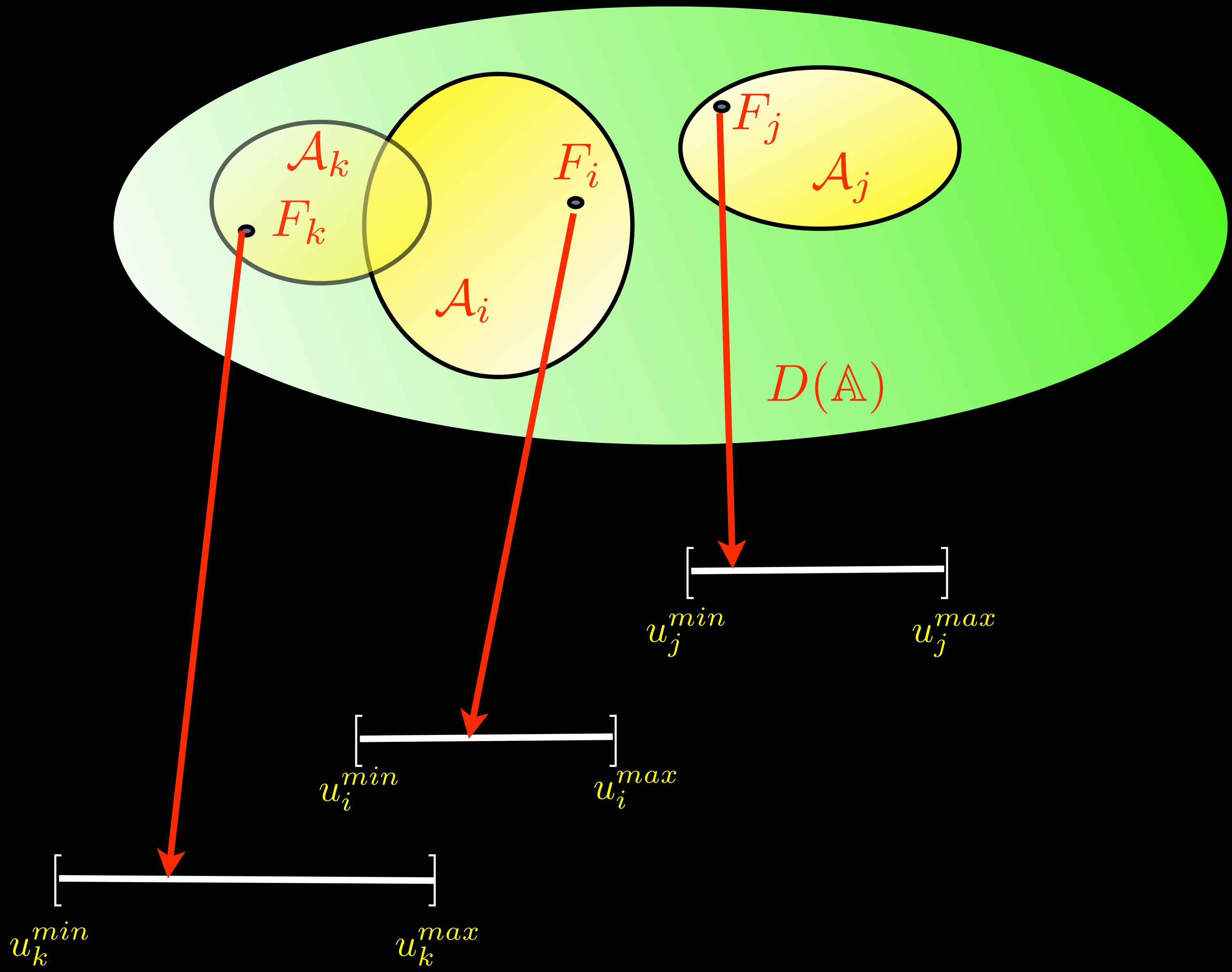
# Separation of sets in $D(\mathbb{A})$.

Suppose that the risk $F_i \in \mathcal{A}_i \subset D(\mathbb{A}), \forall i$ with $\mathcal{A}_i$ compact.

$$u_i^{min} = \min_{F \in \mathcal{A}_i} \{ \int_{\mathbb{A}} u(x) dF(x) \} \qquad u_i^{max} = \max_{F \in \mathcal{A}_i} \{ \int_{\mathbb{A}} u(x) dF(x) \}$$

$\exists I_i = [u_i^{min}, u_i^{max}]$ such that $\int u dF_i \in I_i$






A non-null intersection of intervals implies that alternatives are indistinguishable.

Separating sets becomes complicated when there are more than a few wager alternatives.

Suppose that the number of wager alternatives is integer valued

$G = (V, E)$ is the interval graph associated with $\{I_i; i \in A\}$

An interval graph is an undirected graph such that vertices $v_i, v_j \in V$ are incident if and only if $I_i \cap I_j$ is not empty.

$$i^* = argmax_i\{u_i^{min}\}$$

$I_{i^*}$ is the interval having the greatest lower bound.

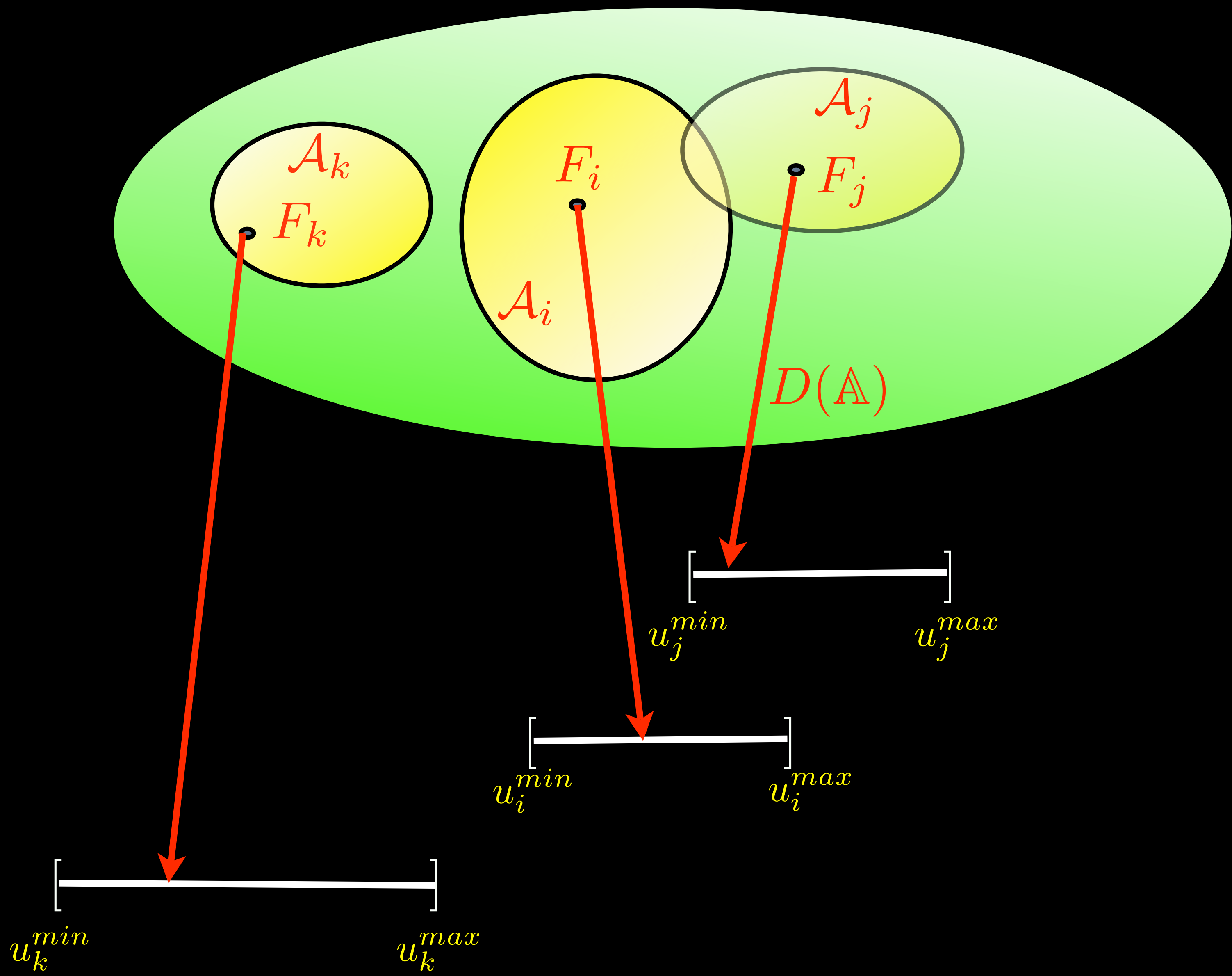
## *Lemma*

*With* $G = (V, E)$ *and* $\{I_i; i \in A\}$ *defined as before,* $v_{i*} \in V$ *belongs to a maximal clique and is incident to no other vertices.*

## *Corollary*

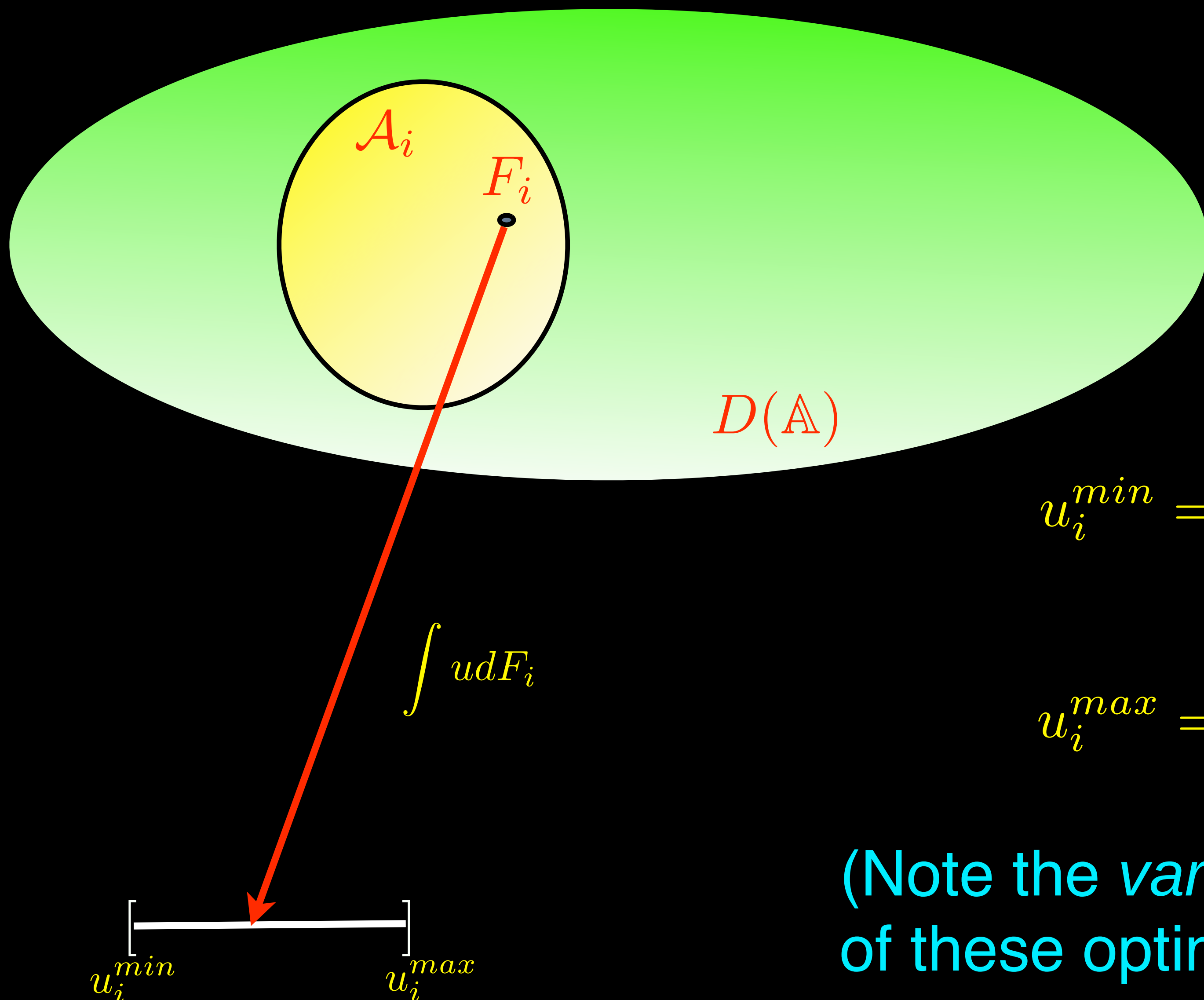
1. *A risk* $F \in \mathcal{A}_i$ *where* $u_i^{max} < u_{i*}^{min}$ *is not preferred to any risk in* $\mathcal{A}_{i*}$.

2. *Any wager alternative* $k$ *for which* $u_k^{max} \geq u_{i*}^{min}$ *is indistinguishable from the most preferred alternative.*




Clearly, alternative $k$ is *not preferred*;
alternatives $i$ and $j$ are *indistinguishable*.

# So - where do we get these sets, $\mathcal{A}_i, i \in A$ ?




$$u_i^{min} = \min_{F \in \mathcal{A}_i} \{ \int_{\mathbb{A}} u(x) dF(x) \}$$

$$u_i^{max} = \max_{F \in \mathcal{A}_i} \{ \int_{\mathbb{A}} u(x) dF(x) \}$$

(Note the *variational* structure form of these optimization formulations.)

First, note that

$$\int_{\mathbb{A}} u(x)dF_i(x) = \int_{\mathbb{R}^k \times \mathbb{R}^k_+} (u \circ V_i)(y)dG_i(y)$$

where

$$G_i(y) = P\{X_1^i \leq y_1, ..., X_k^i \leq y_k, T_1^i \leq y_{k+1}, ..., T_k^i \leq y_{2k}\}$$

and the reward is a finite marked point process

$$\{(X_n^i, T_n^i); n = 0, ..., k\}$$

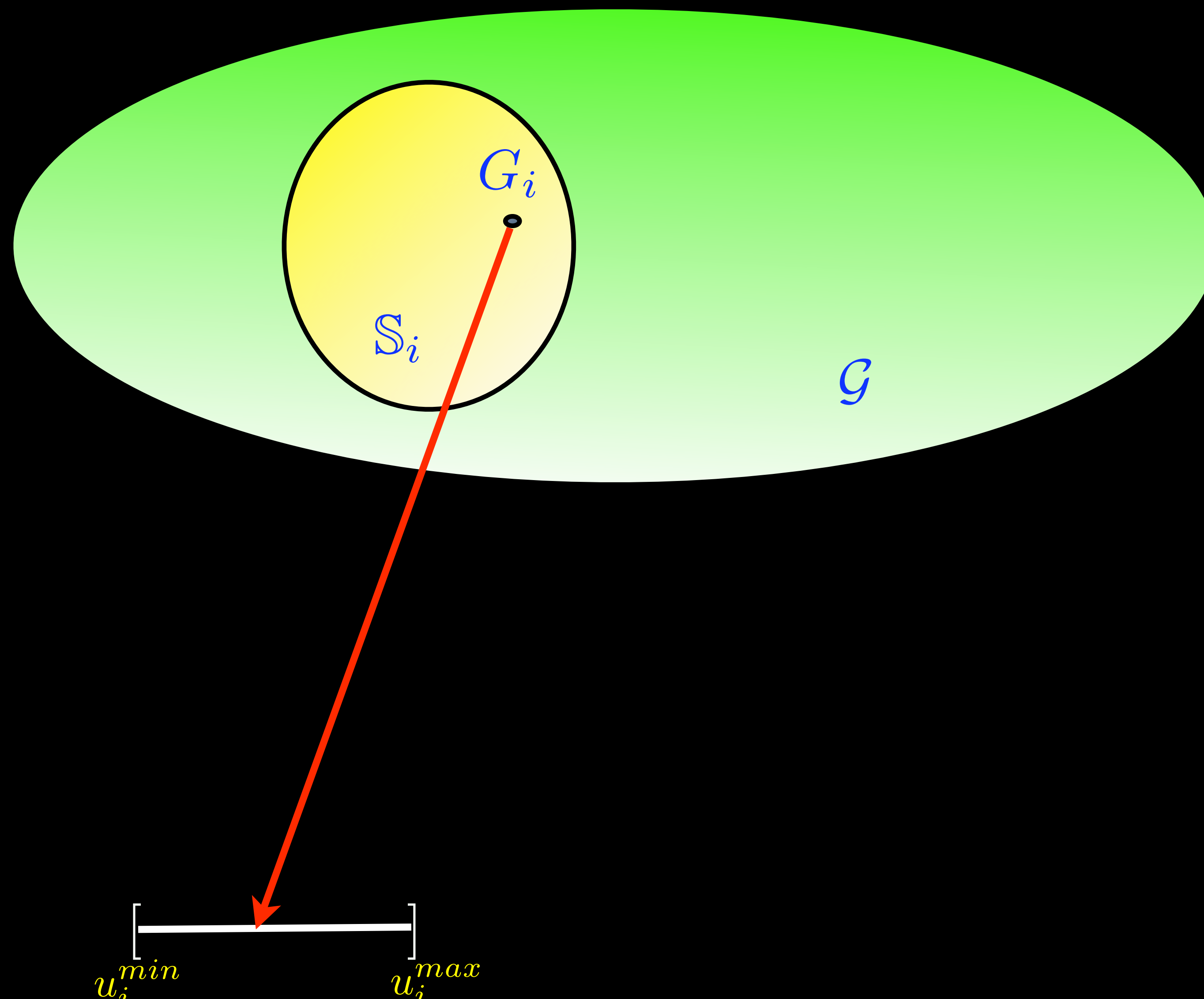
The separating function is simply rewritten in terms of the joint distribution $G_i$ .

$$u_i^{max} = \max_{G \in \mathbb{S}_i} \{ \int_{\mathbb{R}^k \times \mathbb{R}_+^k} (u \circ V_i)(y) dG_{(}y) \}$$

$$u_i^{min} = \min_{G \in \mathbb{S}_i} \{ \int_{\mathbb{R}^k \times \mathbb{R}_+^k} (u \circ V_i)(y) dG_{(}y) \}$$

Two variational formulations for each wager alternative.

How do we solve ‘em?
(With the “heavy artillery.)



# Some Observations:

In engineering, we typically synthesize alternatives, seeking something better. Alternatives cost money.

The value of alternative is not know *a priori* know with certainty. We must gamble by choosing an alternative - seeking most the favorable risk.

It is not (practically) possible to uniquely determine the risk of *any* alternative. We can only identify a set of constraints that risk must satisfy.

We can, however, separate alternatives up to indistinguishability. This requires: 1) solving a (large) number of calculus of variations formulations, and 2) solving a maximal clique formulation.

Engineering wagers can be addressed in familiar computational territory. Modern computing offers the opportunity to overcome many long standing barriers arising in probability models.

# Epilogue:

- The availability of high-performance computational equipment together with quality simulation codes is boon for engineers (gamblers). Simulation codes will form the foundation of predictive modeling.

- Predictive models (probability law on stochastic processes) that rely on large codes are not always easily constructed.

  - Quality of code (verification process) is a serious open issue ... curse of Doob's optional sampling theorem .

  - Validity is determined by the preferences of the gambler ... we must ensure that you don't get garbage out when you don't put garbage in.

- The optimization paradigms characterizing rational gambling are at least as computationally intensive as the subordinate simulation codes.

- Computational probability is in its infancy; much important research remains before it reaches it's promise.